

Further results on Triameter of the Annihilator graph of a commutative ring with identity

*K. Jeya Daisy, V. Sujin Flower and S. Nihisha

Department of Mathematics, Holy Cross College (Autonomous), Nagercoil - 629004

Affiliated to Manonmaniam Sundaranar University, Tirunelveli - 627012

*Corresponding Author - Email: jeядaisy@holycrossngl.edu.in

ABSTRACT

The Annihilator graph of the commutative ring R , denoted by $AG(R)$, is the graph whose vertices are the nonzero zero divisors of the commutative ring. If $a \in Z(R)^*$ then let $ann_R(a) = \{d \in R \mid da = 0\}$, two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. Also, a distance parameter triameter of a connected graph G , which is defined as for $|V(G)| \geq 3$, $tr(G) = \max \{d(u, v) + d(v, w) + d(u, w); u, v, w \in V(G)\}$. In this article we determine the behaviour of the triametrical vertices of $AG(R)$.

Keywords: Distance in graph, Triameter, Annihilator graph, Ring of integer mod n

AMS Subject Classification (2020): 05C12, 13A70

1 Introduction

One of the most rapidly growing areas of mathematics is graph theory. Understanding how to quantify the nodal connections in a network to build communities, is one of the challenges with the clustering process. Investigating the distance between the vertices will help us to find the solution. The diameter and radius of the graph are two of the most frequently seen properties. Angsuman Das [4] first proposed the idea of a distance parameter triameter of a connected graph in 2018, Which is stated as follows: For a connected graph $G = (V, E)$ on $|V(G)| \geq 3$, the distance parameter triameter of G , $tr(G)$ is equal to the maximum of $\{d_G(u, v) + d_G(v, w) + d_G(u, w); u, v, w \in V(G)\}$. The triameter was used as a parameter in [11]. But explicitly named only in [10].

The study of algebraic structures using the properties of graph theory has been an exciting research topic in recent days. The Annihilator graph of the commutative ring R , denoted by $AG(R)$ is the graph whose vertices are the nonzero zero divisors of the commutative ring. If $a \in Z(R)^*$ then let $ann_R(a) = \{d \in R \mid da = 0\}$, two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. This concept was introduced by Ayman Badawi [3] in 2014, For further information see [1, 5]. In this article we determine the behaviour of the triametrical vertices of the annihilator graph $AG(R)$.

We use the following definitions and notations in graph theory [7] in the subsequent sequel. By a graph $G=(V,E)$, we mean an undirected simple graph with vertex set V and edge set E . A graph G is called connected if there is a path between any two distinct vertices in G . for vertices u and v in a graph G , $d(u, v)$ denotes the length of the shortest path from u to v . If there is no such path, then $d(u, v)$ is defined to be infinity; and $d(u, u)$ is defined to be zero. The diameter of a graph G is the supremum of $\{d(u, v) \mid u \text{ and } v \text{ are vertices of } G\}$ denoted by $diam(G)$. A set $D \subseteq V(G)$ is a dominating set if every vertex $v \in V(G)$ is either in D or adjacent to at least one vertex in D . A dominating set with the smallest possible cardinality is minimum dominating set.

Also, we recall some definitions and notations in algebra [9]. We consider R as a commutative ring with identity that is not an integral domain. A ring R in which $ab = ba$ for all $a, b \in R$, is called a commutative ring. A nonempty subset I of a semi ring R is called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. Let R be a commutative ring. let $I \subseteq R$. The annihilator of I is the ideal consisting of the elements $a \in R$ such that $\forall x \in I: ax = 0$. Moreover, a commutative ring R is referred to as reduced if its nilradical, denoted by $Nil(R)$, is zero. Nilradical of R is the set of nilpotent elements. If $Nil(R) \neq 0$ then R is nonreduced. A proper ideal I of R is termed a maximal ideal if whenever J is an ideal of R and $I \subsetneq J \subseteq R$. Further, we use the following theorems

Theorem 1.1: [3] Let R be a commutative ring with $|Z(R)^*| \geq 2$. Then $AG(R)$ is connected and $diam(AG(R)) \leq 2$.

Theorem 1.2: [8] Let R is not a local ring with $|Z(R)^*| > 3$ if and only if $tr(AG(R)) = 6$.

Theorem 1.3: [3] Let R be a commutative ring and x, y be distinct elements of $Z(R)^*$. Then $x - y$ is not an edge of $AG(R)$ if and only if $ann(xy) = ann(x)$ or $ann(xy) = ann(y)$.

Theorem 1.4: [3] If $ann(x) \subseteq ann(y)$ and $ann(y) \subseteq ann(x)$ for some distinct $x, y \in Z(R)^*$, then $x - y$ is an edge in $AG(R)$.

Theorem 1.5: [8] Let R is not a local ring with $|Z(R)^*| > 3$. Let $a, b, c \in Z(R)^*$ be a triametal triple of vertices in the annihilator graph $AG(R)$ then $abc \neq 0$ and $a, b, c \in M_i$ where M_i 's are maximal ideals of R and $i = 1, 2, \dots, n$

2 Main Results

This section we determine the behaviour of the triametal vertices of the annihilator graph $AG(R)$.

Theorem 2.1. Let R be a nonreduced ring and R is not a local ring with $|Z(R)^*| > 3$ then the triametal vertices of $AG(R)$ have at most one nonzero nilpotent element.

Proof. Let R be a nonreduced ring and R is not a local ring. Let a, b and c are the triametal vertices of $AG(R)$. In order to prove that the triametal vertices of $AG(R)$ have atmost one nonzero nilpotent element we assume that the elements a and b are nilpotent. This implies that $a^n = 0$ and $b^m = 0$ where n, m are the positive integers. Then, $(ab)^k = a^k b^k = 0$, where $k = \max(n, m)$. Thus, $\text{ann}(ab) = R$. Moreover, $\text{ann}(a) \subseteq Z(R)$ and $\text{ann}(b) \subseteq Z(R)$. Hence, $\text{ann}(ab) \neq \text{ann}(a) \cup \text{ann}(b)$. By the definition, $d(a, b) = 1$. By Theorem 1.1, $\text{tr}(AG(R)) < 6$. By Theorem 1.2, there is a contradiction that $\text{tr}(AG(R)) = 6$. Therefore, the triametal vertices of $AG(R)$ have atmost one nonzero nilpotent element.

Theorem 2.2. Let R be a nonreduced ring and R is not a local ring with $|Z(R)^*| > 3$ then the triametal vertices of $AG(R)$ have atmost one element which belongs to the intersection of maximal ideals.

Proof. Let R be a nonreduced ring and R is not a local ring. Let a, b and c are the triametal vertices of $AG(R)$. Let $M_1, M_2, \dots, M_n$ be the maximal ideal of R . Assume to the contrary that, $a, b \in M_1 \cap M_2 \cap \dots \cap M_n$. This implies that, $ab = 0$. Thus, $\text{ann}(ab) = R$. Moreover, $\text{ann}(a) \subseteq Z(R)$ and $\text{ann}(b) \subseteq Z(R)$. Hence, $\text{ann}(ab) \neq \text{ann}(a) \cup \text{ann}(b)$. By the definition, $d(a, b) = 1$. By Theorem 1.1, $\text{tr}(AG(R)) < 6$. By Theorem 1.2, there is a contradiction that $\text{tr}(AG(R)) = 6$. Therefore, the triametal vertices of $AG(R)$ have atmost one element which belongs to the intersection of maximal ideals.

Theorem 2.3. Let R is not a local ring with $|Z(R)^*| > 3$. If a_1, a_2 and a_3 are the triametal vertices of $AG(R)$ then $\text{ann}(a_i) \subseteq \text{ann}(a_j)$ where $i \neq j$ and $i, j = 1, 2, 3$.

Proof. Let R is not a local ring with $|Z(R)^*| > 3$. If a_1, a_2 and a_3 are the triametal vertices of $AG(R)$. In order to prove $\text{ann}(a_i) \subseteq \text{ann}(a_j)$ where $i \neq j$ and $i, j = 1, 2, 3$, we assume $\text{ann}(a_1) \not\subseteq \text{ann}(a_2)$. Then by Theorem 1.4, the vertices a_1 and a_2 are adjacent in $AG(R)$. That is, $d(a_1, a_2) = 1$. By Theorem 1.1, $\text{tr}(AG(R)) < 6$. By Theorem 1.2, there is a contradiction that $\text{tr}(AG(R)) = 6$. Therefore, $\text{ann}(a_i) \subseteq \text{ann}(a_j)$ where $i \neq j$ and $i, j = 1, 2, 3$.

Theorem 2.4. Let R is not a local ring with $|Z(R)^*| > 3$ and a_1, a_2 and a_3 are the triametal vertices of $AG(R)$ if and only if $\text{ann}(a_i a_j) = \text{ann}(a_i)$ or $\text{ann}(a_i a_j) = \text{ann}(a_j)$ where $i \neq j$ and $i, j = 1, 2, 3$.

Proof. Let R is not a local ring with $|Z(R)^*| > 3$. If a_1, a_2 and a_3 are the triametal vertices of $AG(R)$. In order to prove $\text{ann}(a_i a_j) = \text{ann}(a_i)$ or $\text{ann}(a_i a_j) = \text{ann}(a_j)$ where $i \neq j$ and $i, j = 1, 2, 3$, we assume that $\text{ann}(a_1 a_2) \neq \text{ann}(a_1)$ and $\text{ann}(a_1 a_2) \neq \text{ann}(a_2)$. This implies that, $\text{ann}(a_1 a_2) \neq \text{ann}(a_1) \cup \text{ann}(a_2)$. By the definition, the vertices a_1 and a_2 are adjacent in

$AG(R)$. That is, $d(a_1, a_2) = 1$. By Theorem 1.1, $tr(AG(R)) < 6$. By Theorem 1.2, there is a contradiction that $tr(AG(R)) = 6$. Therefore, $ann(a_i a_j) = ann(a_i)$ or $ann(a_i a_j) = ann(a_j)$ where $i \neq j$ and $i, j = 1, 2, 3$.

Conversely, assume that $a_1, a_2, a_3 \in Z(R)^*$ such that $ann(a_i a_j) = ann(a_i)$ or $ann(a_i a_j) = ann(a_j)$ where $i \neq j$ and $i, j = 1, 2, 3$. Thus, $ann(a_1 a_2) = ann(a_1)$ or $ann(a_1 a_2) = ann(a_2)$, $ann(a_1 a_3) = ann(a_1)$ or $ann(a_1 a_3) = ann(a_3)$ and $ann(a_2 a_3) = ann(a_2)$ or $ann(a_2 a_3) = ann(a_3)$. By Theorem 1.4, the vertices a_1 and a_2 , the vertices a_1 and a_3 and the vertices a_2 and a_3 are not adjacent in $AG(R)$. This implies that, $d(a_1, a_2, a_3) = 6$. By Theorem 1.2, the vertices a_1, a_2 and a_3 are the triametal vertices of $AG(R)$.

Theorem 2.5. Let R be a commutative ring with identity then the triametal vertices of $AG(R)$ have at least one end vertex if and only if there exists a non zero element $r \in Z(R)^*$ with $|ann(r)^*| = 1$.

Proof. Let $a, b, c \in Z(R)^*$ be the triametal vertices of $AG(R)$. Without loss of generality, let the vertex a be an end vertex in $AG(R)$. Assume to the contrary that for every $r \in Z(R)^*$ with $|ann(r)^*| = 1$. Then every vertex in $AG(R)$ has at least two neighbours, which is a contradiction to the vertex a be an end vertex in $AG(R)$.

Conversely, let there exists a nonzero element $r \in Z(R)^*$ such that $|ann(r)^*| = 1$. On the contrary assume that the triametal vertices of $AG(R)$ do not contain any end vertices. For that let $a, b, c \in Z(R)^*$ be the triametal vertices of $AG(R)$ which were not end vertices. Since $|ann(r)^*| = 1$, there is a unique vertex in $AG(R)$ which annihilates the vertex r . Let it be ' c '. Since the vertex c is not an end vertex and the vertex r does not lie on a path from the vertex a to the vertex c and the vertex b to the vertex c , $d(a, r) = d(a, c) + 1$ and $d(b, r) = d(b, c) + 1$. This implies $d(a, b, r) = d(a, b, c) + 2$. Thus, $d(a, b, r) > d(a, b, c)$, there is a contradiction. Hence, either a, b or c is an end vertex.

Theorem 2.6. Let R be a commutative ring with identity then at most one vertex in the triametal vertices of $AG(R)$ belongs to the minimum dominating set of $AG(R)$.

Proof. Let R be a commutative ring with identity. Let $a, b, c \in Z(R)^*$ be the triametal vertices of $AG(R)$ and D be the minimum dominating set of $AG(R)$. We prove that either a, b or c belongs to D or neither a, b or c belong to D . If R has only one maximal ideal then $AG(R)$ is a complete graph. This implies that at most one vertex in the triametal vertices of $AG(R)$ belongs to D .

Let R has two maximal ideals, namely M_1 and M_2 . If $AG(R)$ is isomorphic to star graph then M_1 has only one vertex and which is the minimum dominating set of $AG(R)$. Thus,

$a, b, c \in M_2$. Hence, neither a, b or c belong to D . Let $AG(R)$ is not isomorphic to star graph. Then D has one vertex of M_1 and one vertex of M_2 . Thus, at most one vertex of a, b, c belongs to D .

Let R has more than three maximal ideals. Without loss of generality let R has three maximal ideals, namely M_1, M_2 and M_3 . Here, R is not a local ring. Then by Theorem 1.5, $a, b, c \in M_i$ where M_i 's are maximal ideals of R . Suppose the vertices a and b belong to the minimum dominating set D . Then $D = \{a, b\}$. We prove this theorem by following two cases. Case (i): Let R is a reduced ring. By Theorem 1.5, let $a, b, c \in M_1$. Then the vertices in $M_1 \cap M_2$ and $M_1 \cap M_3$ are not dominated by D , which is a contradiction to $|D|=2$. Thus, at most one vertex of a, b, c belongs to D .

Case (ii): Let R is not a reduced ring. Let $M_2^n = 0$ for sufficiently large n . By Theorem 1.5, without loss of generality, let $a, b, c \in M_1$. Then the vertices in M_2^{n-1} are not dominated by D , which is a contradiction to $|D|=2$. Thus, at most one vertex of a, b, c belongs to D .

References

1. M. Afkhami, K. Khashyarmansh, Z. Rajabi, Some results on the annihilator graph of a commutative ring, Czechoslovak Mathematical Journal, 67 :1 (2017), 151 - 169.
2. D.F. Anderson, S.B. Mulay, On the diameter and girth of a zero-divisor graph, J. Pure Appl. Algebra, 210 (2007) 543 - 550.
3. A. Badawi, On the Annihilator Graph of a commutative ring, Communications in Algebra, 42:1 (2014), 108 - 121.
4. A. Das, Triameter of graphs, Discuss Math. Graph theory 41 (2021), 601 - 616.
5. S. Dutta, C. Lanong, On the annihilator graphs of a finite commutative ring, Transactions on Combinatorics, 6:1 (2017), 1 - 11.
6. A. Hak, S. Kozerenko and B. Oliynyk, A note on the triameter of graphs, Discrete Applied Mathematics 309 (2022) 278 - 284.
7. F. Harary, Introduction of graph theory, Addition-Wesley publishing company, Inc. 1969
8. K. Jeya Daisy, S. Nihisha, P. Jeyanthi, Triameter of the annihilator graph of a commutative ring with identity, preprint.
9. I. Kaplansky, Commutative rings, Polygonal Publishing House, Washington, NJ, 1974
10. S.R. Kola and P. Panigrahi, A lower bound for radio k -chromatic number of an arbitrary graph, Contrib. Discrete Math. 10 (2015), 45 - 56.
11. L. Saha and P. Panigrahi, Antipodal number of some powers of cycle, Discrete Math. 312 (2012), 1550 - 1557.